Exercise 1. *Give a reduction of 3-COL to 4-COL (or any k-COL) to show that k-COL is NP complete for $k \geq 3$.*

Solution. Given a graph G, consider the graph G' obtained from G by adding a dominating vertex (a vertex with edges to all other vertices). Then G is k-colorable if and only if G' is $(k+1)$-colorable.

Exercise 2. *Complete the reduction from the directed Hamilton cycle problem to the undirected Hamilton cycle problem.*

Solution. Let G be a directed graph. We create G' as follows. For each vertex $v \in V(G)$, we add a path v^-, v, v^+ to G'. For each edge $v \to w \in E(G)$, we add an edge $v^- \to w^+$. Now a Hamilton cycle in G' sees every vertex v by coming in at v^+ and leaving at v^-, corresponding to the incoming and outgoing edges in a directed Hamilton cycle in G. The graph G' has $3n$ vertices and $m + 2n$ edges, so this is a polynomial time reduction in n.

Exercise 3. *A* Hamilton path *is a spanning path. Give a (polynomial time) reduction from the Hamilton cycle to the Hamilton path problem. Then, give an explicit reduction from the Hamilton cycle to the Hamilton path problem. (Note that the latter is not necessary for classification, we have already shown that both problems are NP-complete.)*

Solution. Let G be a graph with an arbitrary vertex v. If G has a Hamilton cycle, the cycle must go through v. For the neighbors $v_1, \ldots, v_k$ of v, let the graphs G_i be the graph G with a leaf w added to v and leaf w_i to v_i. Now G has a Hamilton cycle if and only if at least one of the graphs G_i has a Hamilton path. Indeed, a Hamilton cycle in G must contain an edge of the form vv_i, which corrsponds to a Hamilton path in G_i with endpoints w and w_i (and vice versa). Since there are $O(n)$ neighbors of V, this is a polynomial time reduction.

For the Hamilton path to cycle reduction, let G' be G plus a dominating vertex. It is easy to see that G' has a Hamilton cycle if and only G has a Hamilton path.

Exercise 4. *Write an explicit, polynomial time, algorithm to find augmenting paths, given a graph G and matching M.*

Solution. When we design a variation of a BFS that keeps track of even and odd paths, there are two main pitfalls to avoid. We could discover a vertex twice (once even and once odd) via the same path that crosses itself. Note that this detects an odd cycle in the graph. When trying to resolve this we have to be careful not to accidentally create an algorithm that finds all possible paths between pairs of vertices as that is no longer guaranteed to be polynomial. See Edmonds' Blossom Algorithm, for example.

Exercise 5. *A* vertex cover *in a graph G is a set of vertices S such that every edge has an endpoint in S. In bipartite graphs, the problem of finding a minimum vertex cover is equivalent to finding a maximum independent set. See König's Theorem in Section 2.1 in Diestel. Therefore, for bipartite graphs the vertex cover decision problem is in P. Show that on general graphs, the vertex cover problem is in NP, by reducing from INDEPENDENT-SET.*

Solution. Vertex covers and independent sets are complementary, so they are really the same problem. Since a vertex cover hits every edge in at least one endpoint, its complement must be an independent set, and vice versa.