Exercise 1. *Show that every tree T with root v has an ordering of the vertices $v_1, \ldots, v_n = v$ such that if v_i is a child of v_j in the tree, $i < j$.*

Solution. For simplicity of the algorithm we'll do an order whose reversal satisfies the requirement. First, assign a weak ordering to the vertices by letting their order be the distance from the root. Then, extend this weak order to a linear order. Then every parent-child pair v, w is ordered with $o(v) < o(w)$ due to the weak order, and vertices at equal distance from the root are never parents/children of each other.

Exercise 2. *Prove Claim 1. (We discussed an outline briefly in class.)*

Solution. Suppose that we have a tree decomposition T of width $< n - 1$ for K_n. Not every vertex can appear in every bag, so let v be a vertex which does not appear in every bag. Then there is an edge $B_1 B_2$ in T such that $v \in B_1$ and $v \notin B_2$. As before, we may assume that no bag is a subset of antoher and then there is a vertex w in $B_2 \setminus B_1$. There must exist a bag B that contains both v and w, since they have an edge. However, we have that then either the bag B_1 lies on the path from B to B_2, or B_2 lies on the path from B to B_2. To see this, note that every edge, and therefore the edge $B_1 B_2$, is a bridge, and B lies in either the same component as B_1 or B_2 in $T - B_1 B_2$. Then either the bags containing w or the bags containing v, respectively, do not induce a connected subtree.

Exercise 3. *Prove the following: consider a tree decomposition T of some graph G. For any three vertices $V_i, V_j, V_k \in V(T)$, such that V_j lies on the path between V_i and V_k, we have that $V_i \cap V_k \subseteq V_j$.*

Solution. If a vertex appears in both V_i and V_k, it must appear in every bag on the unique path between them, or else its bags do not induce a connected subtree.

Exercise 4. *Show that the class of series-parallel graphs is closed under taking induced subgraphs. (This is equivalent to showing that this class is closed under vertex-deletion.)*

Solution. First we show that this class is closed under edge deletion. Every edge was created in one of two ways: either it was added when a new vertex was added with one edge to an existing one, or it was created when an edge was subdivided into two edges. In the first case, replace the step where the edge was created by adding a vertex of degree 1 by a step where an isolated vertex is added instead. In the second case, replace the step where the original (pre-subdivision) edge was added by a step where a vertex of degree 1 is added instead, and skip the subdivision step. This vertex replaces the subdivision-vertex.

Now that we can delete edges, deleting vertices instead is simple. For a given series-parallel graph and a vertex to delete, first delete all edges incident to it. Now, we have a series-parallel graph with an isolated vertex. In a construction of this graph, this vertex could only have been added in the step that adds an isolated vertex. Skipping this step deletes the vertex while leaving the graph series-parallel.

Exercise 5. *Give the possible values of n, m such that $P_n \square P_m$ is series-parallel.*

Solution. WLOG, we assume that $n \leq m$. First, we note that when $n = 1$, the grid is a path, which is clearly series-parallel. Now, consider $n = 2$. Build the grid as follows: start with an edge $v_{1,1} v_{2,1}$, and add a parallel edge, which we then subdivide twice, labelling the new vertices as $v_{1,2}, v_{2,2}$. Repeat on the edge $v_{1,2} v_{2,2}$ to create $v_{1,3}, v_{2,3}$, etc.

Suppose that $n > 2$. The grid has 4 vertices of degree 2: the corners. All other vertices have degrees > 2. In a series-parallel construction of the grid, a corner vertex must have been added last, and it must have been added as an edge subdivision. Undoing these 4 subdivisions leaves a graph of minimum degree 3, and therefore this graph could not be series-parallel.