

Exercise 1. Give a set of forbidden induced subgraphs to define the set of complete bipartite graphs.

Solution. One thing to note: in our definition of $K_{n,m}$ here, we must allow m or n to be 0, or else our class is not closed under taking induced subgraphs.

First, we have to forbid all odd cycles, since these graphs are bipartite. If you have not proven this claim before, it might be worth trying to show that graphs contain odd cycles if and only if they contain induced odd cycles. Next, we ensure that if the graph contains an edge, then every other vertex has an edge to one of its two endpoints, by forbidding $K_2 + K_1$.

Exercise 2. Show that the following are equivalent for a graph G :

- (i) G has no subgraph $H \leq G$ such that $\delta(H) \geq 3$,
- (ii) the vertices of G can be ordered v_1, \dots, v_n such that $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| \leq 2$, for $1 \leq i \leq n$.

Solution. Note that if G has an ordering as in (ii), then all of its subgraphs do, since we can simply take the same ordering restricted to the vertices of the subgraph. Then, if a graph G has $\delta(G) \geq 3$, it cannot have an ordering according to (ii), since the final vertex v_n would break the constraint. This shows that (ii) implies (i).

Now, suppose that G satisfies 1. Then G must have a vertex v of degree ≤ 2 . Let this vertex be v_n and proceed by induction. If at any point there is no vertex of degree ≤ 2 this would imply a subgraph of minimum degree ≥ 3 , which is a contradiction.

Exercise 3. Show that $tw(P_n \square P_m) \leq \min(n, m)$.

Solution. WLOG, suppose that $m \leq n$. We use the following tree decomposition T . Let T be a path on bags $B_{i,j}$, $1 \leq i \leq n-1$, $1 \leq j \leq m$, ordered lexicographically along the path, i.e. $B_{1,1}, B_{1,2}, \dots, B_{1,m}, B_{2,1}, \dots$. Let $B_{i,j} = \{v_{i+1,1}, \dots, v_{i+1,j}, v_{i,j}, \dots, v_{i,m}\}$. Each bag has cardinality $m+1$, each vertex $v_{i,j}$ appears first in bag $B_{i,j}$ and stays until bag $B_{i+1,j+j}$. Each edge of the form $v_{i,j}v_{i+1,j}$ appears in bag $B_{i,j}$, and each edge of the form $v_{i,j}v_{i,j+1}$ appears in bag $B_{i,j+1}$ (and others).

Exercise 4. Describe the family of graphs that is $\{K_{1,3}, C_3, C_4, C_5 \dots\}$ -induced-subgraph-free. Then, show that this family is well-quasi-ordered under the induced subgraph relation.

Solution. These graphs are clearly cycle-free, and therefore forests. Then, any vertex of degree ≥ 3 would give an induced $K_{1,3}$ (since there are no cycles). So, these are the forests that consist of paths. Now, we show that this family is well-quasi-ordered under the induced subgraph relation. Suppose that it is not, and we have a “bad” sequence F_1, F_2, \dots . If any F_i contains a path of length at least $2|F_1|$, it would contain F_1 as an induced subgraph (embedding the paths of F_1 into the long path and skipping at least a vertex between each). So, the paths all have bounded length. Let l be the length of a longest path in all of the forests F_1, F_2, \dots , and suppose F_j is the first forest that contains a path of length l , and suppose that it contains k of them. There are either infinitely many forests after F_j that contain kP_l or not. If there are only finitely many, delete them from the sequence and repeat. If there are infinitely many, restrict the sequence to forests that contain kP_l , and then delete kP_l from each (note that this does not affect good/bad pairs). These steps may be repeated until there are no paths of length l left. This holds because $k \leq |F_1|$, since F_1 is a subgraph of $|F_1| \cdot P_l$. Now, repeat on the next longest path, until $l = 1$ and all graphs in the sequence

are empty. This sequence must be good, since the natural numbers are well-quasi-ordered. Since at each step we obtained a sequence that was good if and only if the original sequence was good, we see that F_1, F_2, \dots must have been good; a contradiction.

Exercise 5. *Let \mathcal{F} be the family of graphs of the form $nC_3 + mC_4$. Show that \mathcal{F} is well-quasi-ordered under the induced subgraph relation.*

Solution. Let G_1, G_2, \dots be a bad sequence with $G_i = n_i C_3 + m_i C_4$. We note that a good pair is exactly a pair of the form $i < j$ with $n_i \leq n_j$ and $m_i \leq m_j$. We have seen that if a set is well-quasi-ordered, every sequence contains an infinite increasing subsequence. Since we know that the natural numbers are well-quasi-ordered (repeat this proof for yourself if needed), the sequence n_1, n_2, \dots contains an infinite increasing subsequence n_{k_1}, n_{k_2}, \dots . By the same argument, all we need is that the sequence m_{k_1}, m_{k_2}, \dots contains a good pair k_i, k_j . Then G_{k_i}, G_{k_j} is a good pair in the original sequence; a contradiction.