Graph substructures

We have mostly talk about subgraphs as a substructure of a graph, but there are three variations of interest. We have substructures obtained by:

$$\left.\begin{array}{l}\left.\begin{array}{l}\left.\text{vertex deletion}\right\}\text{ induced subgraph}\\ \text{edge deletion}\end{array}\right\}\text{subgraph}\\ \text{edge contraction}\end{array}\right\}\text{minor.}$$

We say that a class of graphs $\mathcal{F}$ is *closed under* taking induced subgraphs/subgraphs/minors, if $G \in \mathcal{F}$ implies that any such substructure of G is also in $\mathcal{F}$. Graph classes may be finite or infinite. If a graph class is closed under taking some substructure, it can be alternatively defined by what is not in the class: i.e. the forbidden substructures. Often, we don't need to specify all possible substructures not in the class: if $H \leq G$ are both forbidden, then forbidding H implies that G is forbidden. In other words, in the poset of forbidden substructures, it is sufficient to forbid the set of minimal elements.

Example 1. *Consider the set of complete graphs K_n. These are not closed under taking subgraphs (and therefore not closed under minors either), but they are closed under induced subgraphs. We want to forbid any graph with a non-edge, so forbidding $K_1 + K_1$ as an induced subgraph is sufficient.*

Example 2. *Consider the set of bipartite graphs. As we have seen these are the graphs that are odd-cycle free. They are not closed under minors, since an even cycle can be contracted to an odd cycle. We can describe them as the $\{C_3, C_5, C_7, \dots\}$-subgraph-free graphs. Note that $\{C_3, C_5, C_7, \dots\}$ is an infinite anti-chain and therefore no smaller set of forbidden subgraphs exists. As an exercise, you can show that forbidding induced odd cycles also works.*

Exercise 1. *Give a set of forbidden induced subgraphs to define the set of complete bipartite graphs.*

Example 3. *The set of trees is not closed under induced subgraphs, and therefore under none of the 3 substructures. The set of forests, however, is closed under taking minors and therefore all of them. We can describe forests as cycle free graphs, i.e. $\{C_3, C_4, C_5, \dots\}$-(induced)subgraph-free graphs. As before, this an infinite anti-chain. However, all cycles have C_3 as a minor, and therefore forests are the K_3-minor-free graphs.*

Example 4. *The set of planar graphs are the graphs that can be embedded (drawn) on the plane/sphere (a piece of paper) without crossing edges. See Chapter 4 in Diestel [1] for a topologically more precise definition. By Wagner/Kuratowski, the planar graphs are exactly the $\{K_5, K_{3,3}\}$-minor-free graphs.*

Example 5. *As one more example of a class of graphs that is only induced-subgraph free, consider the* co-graphs*. Co-graphs are defined as follows. K_1 is a co-graph. If G and H are two co-graphs, then $G + H$ (the disjoint union) and $G \nabla H$ (the join: the disjoint union plus all edges of the form v, w with $v \in V(G)$ and $w \in V(H)$). It turns out that co-graphs are exactly the class of P_4-induced-subgraph-free graphs.*

Planar graphs have unbounded treewidth

Finally, we return back to the series-parallel graphs, with a simple forbidden-substructure characterization.

Theorem 6 (Duffin '65 [2]). *A graph G is series-parallel if and only if it has no minor isomorphic to K_4.*

In class, we discussed a few steps and an overview of the proof for this theorem, but we will not include the full proof here. As an exercise, the following step is helpful towards seeing the connection.

Exercise 2. *Show that the following are equivalent for a graph G:*

(i) G has no subgraph $H \leq G$ such that $\delta(H) \geq 3$,

(ii) the vertices of G can be ordered $v_1, \ldots, v_n$ such that $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leq 2$, for $1 \leq i \leq n$.

To prove Theorem **??**, one can show that property (i) is equivalent to G having a K_4 minor, and property (ii) is equivalent to G being series-parallel.

We now have that the class of graphs of treewidth ≤ 1 are the K_3-minor-free graphs, and the class of graphs of treewidth ≤ 2 are the K_4-minor-free graphs. This sentence, with the fact that $\text{tw}(K_n) = n - 1$, is suggestive of a conjecture, but that conjecture is false. Recall that planar graphs do not have K_5-minors. However, we will now show that planar graphs can have arbitrarily high treewidth. Specifically, we will show that large square grids have large treewidth. In order to do so, we use the following helpful lemma.

Lemma 7. *If $\text{tw}(G) = k$, then there exists a set $S \subseteq V(G)$, such that $G - S$ has connected components $S_1, S_2, \ldots, S_t$ with $|S_i| < \frac{n}{2}$ for all $1 \leq i \leq t$.*

Proof. Suppose that $k + 1 < \frac{n}{2}$. Otherwise, the results holds trivially. First, we recall that in a tree decomposition T of a connected graph G, each internal bag B represents a cut-set of G: separating the parts of G represented by the subtrees of $T - B$. Of course, not every bag does so in a balanced manner. Let T be rooted at a vertex B (chosen arbitrarily). We will say that a bag B_i is "heavy" if the subtree rooted at B_i represents $< \frac{n}{2}$ of the vertices, and "light" otherwise. Note that every heavy bag has at most one heavy descendant. Therefore, the heavy bags form a path starting at the root. Let B' be the last heavy bag on this path, and T' the subtree rooted at B'. Since B' is heavy, $T - T'$ is light. Furthermore, all subtrees rootes at B's children are light. Then, B' satisfies the Lemma. □

Theorem 8. *We have $\text{tw}(P_n \square P_m) \to \infty$ as $\min(n, m) \to \infty$.*

Proof. We will show that $\text{tw}(P_n \square P_n) \geq \frac{n}{2} - 1$. Suppose that $\text{tw}(P_n \square P_n) = k < \frac{n}{2} - 1$. Then there exists a cut set S with $|S| < \frac{n}{2}$ that satisfies Lemma 7. The at least $\frac{n}{2}$ columns of the grid do not intersect with S, and since at least $\frac{n}{2}$ rows also do not intersect with S, we have a connected subgraph containing at least $\frac{n}{2}$ of the vertices. This contradicts the assumption that S is a cut set satisfying Lemma 7. □

In fact, we have $\text{tw}(P_n \square P_m) = \min(n, m)$, but this is a bit harder to show, and not as important as the fact that we have unbounded treewidth in this class.

Exercise 3. *Show that $\text{tw}(P_n \square P_m) \leq \min(n, m)$.*

Graph substructures and well-quasi-ordering

For this section, we follow the notation in Chapter 12 of Diestel [1]. A *quasi-ordering* is a binary relation on a set that is reflexive and transitive. A quasi-ordering on X is a *well-quasi-ordering* if for every infinite sequence $x_0, x_1, \ldots$ in X there exists a pair i, j such that $i < j$ and $x_i \leq x_j$. We call (x_i, x_j) a *good pair* and a sequence containing a good pair a *good sequence.*

Proposition 9 (Prop. 12.1.1 in [1])**.** *A quasi-ordering $\leq$ on X is a well-quasi-ordering if and only if X contains neither an infinite antichain nor an infinite strictly decreasing sequence $x_0 > x_1 > \ldots$.*

Corollary 10 (Cor. 12.1.2 in [1])**.** *If X is well-quasi-ordered, then every infinite sequence in X has an infinite increasing subsequence.*

For a set X, we let $[X]^{<\omega}$ denote the set of all finite subsets of X.

Lemma 11 (Lem. 12.1.3 in [1])**.** *If X is well-quasi-ordered by $\leq$, then so is $[X]^{<\omega}$.*

Exercise 4. *Describe the family of graphs that is $\{K_{1,3}, C_3, C_4, C_5 \ldots\}$-induced-subgraph-free. Then, show that this family is well-quasi-ordered under the induced subgraph relation.*

Exercise 5. *Let $\mathcal{F}$ be the family of graphs of the form $nC_3 + mC_4$. Show that $\mathcal{F}$ is well-quasi-ordered under the induced subgraph relation.*