

This week we will begin with talking a bit about bipartite graphs from Notes 1, which will then lead us into Hall's Theorem.

Hall's Theorem

Suppose that we have a set of n students and a set of at least n sandwiches, each with different ingredients. Each student lists the sandwiches that they would be happy to eat (based on preference, dietary restrictions, etc.). Is it possible to distribute the sandwiches so that everyone is happy? Suppose that a subset X of the students is interested only in a subset Y of the sandwiches such that $|Y| < |X|$. For example, perhaps there are two students who have only listed the same single sandwich. This is clearly a problem that prevents everyone being happy. As it turns out, this is the only problem we need to worry about.

Theorem 1 (Hall's Theorem.). *Let G be a bipartite graph with partite sets A and B . We have that $|N(S)| \geq |S|$ for every subset $S \subseteq A$ if and only if G contains a matching of A to B .* 2.1.2, p.38

In the above, the notation $N(S)$ indicates the neighborhood of the set S , meaning the set of all vertices in $V(G)$ that have at least one neighbor in S . In class, we will go over the second proof in Diestel for Hall's Theorem first. Then, we will talk about augmenting paths, both for the first proof of Hall's theorem and for König's theorem.

Exercise 1. *Show that if a bipartite graph G is k -regular, then it satisfies Hall's condition.*

Exercise 2. *Let M be a matching in a bipartite graph G . Show that if M is sub-optimal (contains fewer edges than some other matching in G), then G contains an augmenting path with respect to M . Does the result generalize to non-bipartite graphs?*

Exercise 3. *Let G be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$. Suppose that G has a matching M_1 that covers $X_1 \subseteq V_1$ and a matching M_2 that covers $X_2 \subseteq V_2$. Show that G has a matching that covers $X_1 \cup X_2$.*

Exercise 4. *Let M be a matching in a simple graph G . Show that G has a maximum matching that covers all of the vertices covered by M . Hint: over all the maximum matchings in G , consider one with the highest number of edges in common with M .*

König's Theorem is an important example of a type of min/max result that we will see a couple of times in this course. Minimizing one type of structure is equivalent to maximizing another. This result is closely related to Hall's Theorem, and Menger's Theorem and the Min-cut Max-flow Theorem.

Theorem 2 (König's Theorem.). *If G is a bipartite graph, then the maximum cardinality of a matching is equal to the minimum cardinality of a vertex cover of its edges.*

Exercise 5. (★) *The proof of König's Theorem in the book is quite dense. Write your own version of the proof with a bit more explanation of some of the details.*

Exercise 6. *Over all connected graphs on n vertices, what is the lowest value that the size of a minimum vertex cover can take (in terms of n)? What is the highest?*

For both cases, give an example of a graph with a lowest/highest value, and argue why there are no graphs with lower/higher values.