

k -connectedness

We say that a graph G is k -connected (or, k -vertex-connected) if $G \sim K_n$ or if G is connected and remains connected if the deletion of any set of at most $k - 1$ vertices leaves G connected. In other words, disconnecting G requires the removal of at least k vertices. Note that this generalizes the definition of connectedness that you are already familiar with, which is really 1-vertex-connectedness. We will start with investigating the structure of 2-connected graphs. Let H be a subgraph of a graph G . We call a path P in G an H -path if the endpoints of P lie in H , but all other vertices of P are outside of H . We will also call this an *ear* on H . Now, we let an *ear decomposition* be a series of subgraphs of G : $C = H_0, H_1, \dots, H_k = G$, such that C is a cycle in G , and each H_i is obtained from H_{i-1} by attaching an ear.

Proposition 1. *Every 2-connected graph has an ear decomposition.*

3.1.1 p.59

We can find a similar result for edge-connectedness. We say that a graph G is k -edge-connected if G is connected and remains connected if the deletion of any set of at most $k - 1$ edges leaves G connected.

We call a *closed ear* on H a cycle that has exactly one vertex in common with H .

Proposition 2. *Every 2-edge-connected graph has an ear decomposition that uses open and/or closed ears.*

We start a few more exercises to familiarize ourselves with vertex- and edge-connectivity. We say that G has vertex-connectivity $\kappa(G)$ if G is κ -connected but not $\kappa + 1$ -connected. Similarly, we label the edge-connectivity as $\lambda(G)$.

Exercise 1. *Show that $\kappa(G) \leq \lambda(G)$ for every graph G .*

Exercise 2. *Show that $\lambda(G) \leq \delta(G)$ for every graph G .*

Exercise 3. *Suppose that G has all vertices of even degree. What do we know about the parity of $\kappa(G)$? What do we know about the parity of $\lambda(G)$?*

Block decomposition

We have talked about connected components of graphs. If a graph is not connected, then it has multiple such components. Intuitively, it is easy to understand what those are. Here is a more formal definition: For $G(V, E)$ a graph, a *connected component* of G is a maximal connected subgraph. In other words, if H is a connected component of G , then H is a connected subgraph of G and there is no other connected subgraph H' such that $H < H'$ (H is not a proper subgraph of any connected subgraph H'). From this definition, we can replace the word connected with 2-connected. Such subgraphs are blocks. However, we would like to be able to fully decompose a graph G into its blocks, just like we did with the connected components. A *decomposition* of a graph G is a set of subgraphs $H_1, \dots, H_k$ of G and whose union is G , and such that $E(H_1), \dots, E(H_k)$ is a partition of $E(G)$.

So, we allow a few more substructures to be called blocks. Formally, a *block* is a maximal connected subgraph without a cut-vertex. Now, G is the union of its blocks, and every edge lies in a unique block.

Exercise 4. *Show that the blocks of G are indeed a graph decomposition.*

Lemmas 3.1.2 and 3.1.3 show us that block decompositions are a natural way to understand cycles and cuts in connected graphs. First, we make an observation about minimal disconnecting edge sets (sets of edges $S \subseteq E(G)$ such that $G - S$ is disconnected). For two subsets of vertices A and B , we let

$$[A, B] := \{ab \in E(G) \mid a \in A, b \in B\}.$$

We let $G[A]$ denote the subgraph of G induced by A . This is the subgraph of G whose vertex set is A and whose edge set is all edges of G that have both endpoints in A .

Claim 3. *A minimal disconnecting edge set in a connected graph G is of the form $[A, \bar{A}]$, where $A \subseteq V(G)$ and $\bar{A} = V(G) \setminus A$. Furthermore, $G[A]$ and $G[\bar{A}]$ are both connected.*

Proof. Let S be a minimal disconnecting edge set. Then $G - S$ has multiple connected components. For an arbitrary component G_1 of $G - S$, note that all edges in S that have exactly one endpoint in G_1 form an edge cut. Since S is minimal, we must have that all edges of S have exactly one endpoint in G_1 . Therefore, we let $A = V(G_1)$. All that remains to show is that $G[\bar{A}]$ is connected (i.e. that $G - S$ has exactly two connected components). Finish this proof in the next exercise. $\square$

Exercise 5. *Finish the proof of Claim 3.*

Now, we call an *edge cut* of a graph G a set of edges S that is of the form $[A, \bar{A}]$ for some $A \subseteq V(G)$, and a *bond* is a minimal nonempty edge cut. Note that bonds exist even for disconnected graphs.

Lemma 4. *Let G be any graph.*

3.1.2, p.60

- (i) *The cycles of G are precisely the cycles of its blocks.*
- (ii) *The bonds of G are precisely the bonds of its blocks.*

Lemma 5. *Let G be any graph. The following statements are equivalent for any $e, f \in E(G)$:*

3.1.3, p.61

- (i) *The edges e, f belong to common block of G .*
- (ii) *The edges e, f belong to common cycle of G .*
- (iii) *The edges e, f belong to common bond of G .*

If A is the set of cutvertices of G , and $\mathcal{B}$ the set of blocks, then we have a natural bipartite graph on $A \cup \mathcal{B}$ formed by edges aB if $a \in B$. The proof of the following lemma is left as an exercise.

Lemma 6. *The block graph of G forms a tree.*

3.1.4, p.61

Menger's Theorem

Menger's theorem "connects" the idea of connectivity as discussed above with the idea of redundancy in the sense of robustness. Intuitively, a well-connected graph should have multiple paths between two vertices making it robust to some deletions. Section 3.3 has two versions of Menger's Theorem, one for connectivity between sets and one for connectivity between individual vertices.

Theorem 7. *For a graph G with $A, B \subseteq V(G)$ (not necessarily disjoint), the minimum number of vertices separating A from B is equal to the maximum number of disjoint $A - B$ paths in G .* 3.3.1, p.67

Theorem 8. *A graph G is k -connected if and only if it contains k internally disjoint paths between any pair of vertices. A graph G is k -edge-connected if and only if it contains k edge-disjoint paths between any pair of vertices.* 3.3.6, p.72

Exercise 6. (★) *Suppose that a graph G is 2-connected. Let P be a vw -path in G for some pair of vertices $v, w \in V(G)$. Must there exist another vw -path P' that is internally disjoint from P ?*

Exercise 7. (★) *Use Menger's Theorem to prove Hall's Theorem. (You only need to prove the sufficiency of Hall's Condition, not the necessity.)*