

Edge colorings

Just as with vertices, we can ask about proper colorings of edges (or in other words, partitions of the edge set of a graph into matchings). We say that a *proper k -edge-coloring* of a graph G is a function $c : E(G) \rightarrow \{1, \dots, k\}$ such that $e \cap f \neq \emptyset$ implies that $c(e) \neq c(f)$. We let $\chi'(G)$ denote the *chromatic index* or *edge-chromatic number* of G , which is the smallest value of k such that a proper k -edge coloring is possible. Just as with the vertex colorings, we find two lower bounds based on largest clusters of edges (edges incident to a single vertex) and largest independent sets of edges (matchings). Let $\alpha'(G)$ denote the size of a largest matching in G . We have that

- $\Delta(G) \leq \chi'(G)$
- $\frac{|E(G)|}{\alpha'(G)} \leq \chi'(G)$.

Exercise 1. Suppose that we try a greedy algorithm similar to the way we started with vertex colorings. Order the edges $e_1, \dots, e_m$, and color each edge in turn with the lowest available color (lowest color not already used by an edge incident to it). What upper bound on $\chi'(G)$ do we obtain from this approach?

As it turns out, edge colorings behave quite differently from vertex colorings. The “easy” lower bound $\Delta(G) \leq \chi'(G)$ is very close:

Theorem 1 (Vizing’s Theorem). *We have that*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Thm 5.3.2
p.128

Just because the chromatic index is known to be one of two values, it is still difficult to compute exactly. In other words, the class of graphs for which $\chi'(G) = \Delta(G)$ is not easily characterized. We do know that it contains all bipartite graphs. Graphs which achieve the lower bound of Vizing are often called *Type I* and graphs which achieve the upper bound of Vizing are often called *Type II*.

Proposition 2 (König). *We have that, if G is bipartite,*

$$\chi'(G) = \Delta(G).$$

Thm 5.3.1
p.127

We talked about the proof for the latter proposition in class. Try to read this proof and understand it well. Then, you will notice why it fails on general graphs. The proof for Vizing’s Theorem uses a couple of tricks to get around this issue. The tricks should be reminiscent of the proof of the 5-color Theorem for vertex-coloring planar graphs.

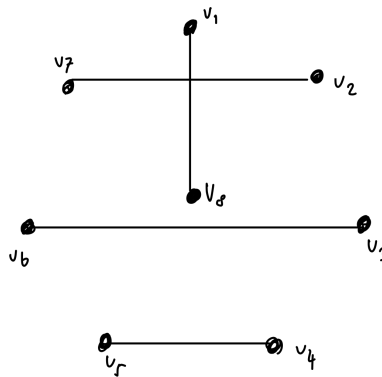
The chromatic index of the complete graphs

We show that

$$\chi'(K_n) = \begin{cases} n - 1, & \text{when } n \text{ is even,} \\ n, & \text{when } n \text{ is odd.} \end{cases}$$

Observation 3. *If G is k -regular, then in any k -edge-coloring of G (if it exists) every color class forms a perfect matching.*

This is easy to see, since every vertex must have (exactly) one edge of every color incident to it. We immediately conclude that regular graphs of type I must have an even number of vertices, and therefore K_n is of type II for odd n . For n is even, and K_n on vertex set $v_1, \dots, v_n$ we provide the following, geometric coloring. Place vertex v_n on the plane, and vertices $v_1, \dots, v_{n-1}$ spaced evenly around a circle centered at v_n . For each $1 \leq i \leq n-1$, color the edge $v_n v_i$ and the edges v_{i+k}, v_{i-k} for $1 \leq i \leq \frac{n}{2} - 1$ with color i . In other words, color 1 looks like the picture below, and each other color is a rotation of this picture.



Exercise 2. Let n be odd. Show that you must delete at least $\frac{n-1}{2}$ edges from K_n to obtain a subgraph of chromatic index $n - 1$. Then, show that this number is indeed sufficient.

Exercise 3. Denote the vertices of K_8 by the bit-strings $\{0, 1\}^3$. Now, find a new proper 7-coloring of its edges, and show that it is distinct (aside from vertex/color relabellings) from the coloring obtained by the geometric method above.

Exercise 4. Suppose that G is a 3-regular graph on an even number of vertices, and that G has a Hamilton cycle. (A cycle in G that uses all vertices.) Show that $\chi'(G) = \Delta(G) = 3$.

Exercise 5. An $n \times n$ matrix with entries from $\{1, \dots, n\}$ is called a Latin square if every element of $\{1, \dots, n\}$ appears exactly once in each column and once in each row. Rephrase the problem of finding Latin squares as a graph coloring problem.