

Extremal graph theory

Extremal questions are generally of the form: how large/small can a combinatorial object be while maintaining a certain property? We explore two main pillars of this style of research: Ramsey and Turán theory.

Ramsey theory

Ramsey theory asks: if we fix the number of vertices, then a high density of edges guarantees the appearance of certain subgraphs, while a low density of edges guarantees subgraphs in the complement of the graph, so when are certain structures guaranteed to appear in one or the other? Instead of thinking of graphs and their complements, we can think of taking the complete graph K_n and coloring the edges red/blue. The questions are now of the form: what is the smallest number of vertices such that every red/blue coloring of the edges of K_n has either a red copy of G or a blue copy of H . In 1930, Ramsey proved that such a number exists.

Theorem 1. *For every $s, t \in \mathbb{N}$, there exists an n such that every red/blue coloring of the edges of K_n has either a red copy of K_s or a blue copy of K_t .* Thm 9.1.1 p.284

We let $R(s, t)$ be the smallest such n , and let $R(s) = R(s, s)$. We did most of the following proofs in class, and I ask you here to write them out fully in your own words.

Exercise 1. *Write the full proof that $R(3) = 6$. (Lower and upper bound.)*

Exercise 2. *Show that*

$$R(s, t) \leq R(s - 1, t) + R(s, t - 1).$$

Exercise 3. *Show that $R(3, 4) \leq 9$.*

Optional Exercise. *Show that $R(3, 4) = 9$.*

Turán theory

Now, we switch to just looking for subgraphs in the edges present in a host graph, and ask for the largest size of a graph, in terms of its edges, that avoids given forbidden subgraphs. We denote by $ex(n, F)$ the largest number of edges in a graph on n vertices which does not contain F as a subgraph.

As a warmup example in class, we will prove Mantel's Theorem, which was the precursor of the famous Turán's Theorem. Although it is called Mantel's Theorem, the proof is due to Wijnthoff. (Mantel wrote the problem for a publication of the Dutch Royal Math Society, and Wijnthoff's submitted solution was chosen for publication.)

Theorem 2. *We have*

$$ex(n, K_3) = \left\lceil \frac{n}{2} \right\rceil \times \left\lfloor \frac{n}{2} \right\rfloor,$$

and the unique graph that achieves this maximum is $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

Later, Turán generalized this result to $ex(n, K_t)$, by showing that the unique maximum graph is the complete $(t - 1)$ -partite graph with partite sets as balanced as possible. We call this the Turán graph T_{t-1} .

Theorem 3 (Turán). *We have*

$$ex(n, K_t) = ||T_{t-1}||,$$

Thm 7.1.1
p.175

and T_{t-1} is the only graph that achieves this maximum number of edges. This implies that

$$\frac{ex(n, K_t)}{\binom{n}{2}} \rightarrow \frac{t-2}{t-1}, \text{ as } n \rightarrow \infty.$$

Exercise 4. *Complete at least two of the following questions.*

- (a) Find $ex(n, P_3)$, where P_3 is the path on 3 vertices.
- (b) Find $ex(n, P_4)$, where P_4 is the path on 4 vertices.
- (c) Find $ex(n, K_{1,t})$.
- (d) Find $ex(n, 2K_2)$.

Exercise 5. *Find $ex(n, G)$, where G is a K_3 with an extra leaf attached. (Or, the graph obtained from K_4 by deleting two incident edges.)*

The previous exercise is a small hint towards the Erdős-Stone theorem, which says that asymptotically, the chromatic number of a graph determines its extremal numbers and graphs. This theorem resolves the extremal numbers asymptotically for all graphs except bipartite graphs.

Theorem 4 (Erdős-Stone). *Let H be a graph with chromatic number $\chi(H) = t$. We have*

$$\frac{ex(n, H)}{\binom{n}{2}} \rightarrow \frac{t-2}{t-1}, \text{ as } n \rightarrow \infty.$$

Thm 7.1.2
p.178